

## Minimum Error Dispersion Linear Filtering of Scalar Symmetric Stable Processes

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**Abstract**—The well-known Kalman-Bucy linear-filtering theory for Gaussian Markov processes is generalized to cover a particular class of non-Gaussian Markov processes, the scalar symmetric stable Markov processes. Results are presented only for discrete time because of certain pathologies that arise in the continuous-time analog (except in the Gaussian case). Attention is confined to the scalar case because of technical problems arising in characterizing multivariate stable distributions (except in the Gaussian case).

### I. INTRODUCTION

Since the original work by Wiener [1], [2], using spectral factorization techniques, and Kalman and Bucy [3], [4], using time domain methods, linear filtering of Gaussian Markov random processes with rational spectra has been extensively studied in both the engineering [5] and information-theoretic [6] literature.

Here we deal with a natural extension of the Kalman-Bucy technique to linear filtering of a particular class of scalar Markov random processes that include the Gaussian as a special case, the symmetric stable random processes in discrete time. The motivation for this work is drawn from problems associated with noise on telephone lines [7], extremely low-frequency electromagnetic communication [8], and the random-walk model of stock-market prices [9], [10].

One reason for the popularity of Gaussian random processes in modeling uncertainty is that often a large number of independent disturbances contribute to the uncertainty in a measurement, and at that point the Central Limit Theorem of probability theory is invoked to claim that the noise is Gaussian. It is often overlooked that the Gaussian is only one member of a family of distributions, the stable distributions, that arise from examining the limiting distributions of suitable normalized and translated sums of random variables. In an engineering situation, one would like to be able to combat uncertainty in a manner independent of the choice of a particular stable distribution.

Two main reasons have contributed to attention being confined to the Gaussian rather than non-Gaussian stable case. First, only in the Gaussian case does there exist a closed-form analytic expression for the probability density function; in the non-Gaussian stable case one must resort to power series and asymptotic series. However, with the availability of digital computers that are inexpensive to use, numerical calculations of error probabilities are straight-forward to carry out. The second main reason that non-Gaussian stable distributions have not been studied or used as widely as Gaussian stable distributions is that only the Gaussian distribution has a finite second moment, and hence a finite variance, and infinite variance is felt to be physically inappropriate. While superficially appealing, this reason is inadequate because the Gaussian distribution is unbounded, which is clearly physically inappropriate. The question here is over what range the model should fit observed data: it may well be that a finite-variance Gaussian model is

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adequate over a limited range of data, while an infinite-variance stable-distribution model is adequate over a larger range of data, but both models are inappropriate outside of a given range of observations, although the infinite-variance model is better than the finite-variance model in terms of matching observed data over a wider range. For sufficiently large deviations from the model nonlinearities and limiting operations would become significant and make either model inappropriate.

Three items are particularly interesting about these results.

1) At the present time, no analogous Wiener-Hopf spectral factorization theory exists for the problems considered here. Hopefully, they will provide stimulus for clarifying the role played by Hilbert space orthogonal projection techniques [1], [2], [6].

2) The generalization to the multivariate case is nontrivial because of the complicated nature of multivariate stable distributions [12]; an Appendix gives the reader who is not familiar with stable distributions a brief summary of their properties in the scalar case. Only in the Gaussian case is it seemingly straightforward to extend these results to multivariate distributions. Because of this we deal only with the scalar case.

3) The extension of these results to continuous time results in a degenerate problem, because of certain pathologies in the sample paths of stable processes [11]. In particular, we can discriminate perfectly between two stable processes with differing parameters in any finite time interval because the associated path space probability measures are orthogonal [11]. The intuitive reason for this is that continuous-time stable processes have sample paths that are left continuous with right-hand limits everywhere defined and are constant except for jumps of random amplitudes at random time epochs; however, the jump time epochs, while countable in any finite time interval, are dense, and thus there is a countably infinite set of jumps available for processing. It may not be surprising that with an infinite amount of data, i.e., observations of the jump epochs and jump amplitudes, perfect filtering and discrimination is possible. In the Gaussian case the sample paths consist of jumps of zero amplitude, and perfect filtering and discrimination is not possible in all cases. This reasoning is only meant as a plausibility argument to suggest why attention is confined here solely to the discrete time case; the diligent reader should consult the references for the detailed technical arguments substantiating these claims [11].

## II. THE RESULT

The reader is referred to the Appendix for a brief discussion of properties of stable distributions. The Appendix also fixes the notation and nomenclature used throughout this note.

Let  $\{u_k\}$ ,  $\{w_k\}$ ,  $k=0,1,\dots$ , be sequences of mutually independent stable random variables with the same characteristic exponent  $\alpha, 0 < \alpha < 2$ , but differing dispersion parameters, denoted by  $\gamma_w$  and  $\gamma_u$ , respectively, where  $k$  is the discrete time parameter.

$$E(e^{i\omega x_k}) = \exp(-\gamma_u |\omega|^\alpha) \quad E(e^{i\omega w_k}) = \exp(-\gamma_w |\omega|^\alpha), \quad k=0,1,\dots$$

Here we have explicitly assumed  $\{u_k\}$ ,  $\{w_k\}$  are sequences of symmetric stable random variables. The dispersion plays the same role as the variance in the Gaussian case, and thus  $\gamma^{1/\alpha}$  plays the same role as the standard deviation in the Gaussian case. For example, if  $x$  is a stable random variable with dispersion  $\gamma$ , then if we scale  $x$  by  $\gamma^{1/\alpha}$ , we see that with location parameter zero,

$$E[\exp(i\omega x \gamma^{-1/2})] = \exp(-|\omega|^\alpha)$$

and hence the natural units of  $x$  are in terms of  $\gamma^{1/\alpha}$ . For simplicity of exposition, we omit the case of asymmetric or skewed stable distributions.

The model of the signal process  $\{x_k\}$  and observation process  $\{y_k\}$  is assumed to be of the form

$$x_{k+1} = Ax_k + Bu_k \quad E(e^{i\omega x_k}) = \exp(-\gamma_x |\omega|^\alpha) \quad y_k = Cx_k + w_k.$$

This encompasses both autoregressive and moving average models, which are currently quite popular in applications.

We now closely follow the original Kalman-Bucy [3], [4] work. We seek a linear signal estimate using a recursive algorithm such that the error sequence has minimum dispersion for each value of time. Thus, we denote by  $\{\hat{x}_k\}$  the signal estimate sequence, and  $\{e_k \equiv x_k - \hat{x}_k\}$  denotes the error sequence.

### Proposition

Given the preceding assumptions, the minimum error-dispersion linear signal-estimate recursive sequence and error sequence satisfy the following relations:

$$\hat{x}_{k+1} = A\hat{x}_k + G_k(y_k - C\hat{x}_k) \quad E(e^{i\omega e_k}) = \exp(-\gamma_{e_k} |\omega|^\alpha)$$

$$e_{k+1} = (A - G_k C)e_k + Bu_k - G_k w_k$$

$$E(e^{i\omega e_k}) = \exp(-\gamma_{e_k} |\omega|^\alpha), \quad \gamma_{e_0} = \gamma_{x_0} + \gamma_{w_0}$$

where for  $1 < \alpha < 2$

$$G_k = \frac{A}{C} \frac{[|C|^{\alpha} \gamma_{e_k}]^{1/(\alpha-1)}}{[|C|^{\alpha} \gamma_{e_k}]^{1/(\alpha-1)} + \gamma_w^{1/(\alpha-1)}}$$

$$\gamma_{e_{k+1}} = |B|^{\alpha} \gamma_u + \left| \frac{A}{C} \right|^{\alpha} \frac{\gamma_w |C|^{\alpha} \gamma_{e_k}}{[\gamma_w^{1/(\alpha-1)} + \{|C|^{\alpha} \gamma_{e_k}\}^{1/(\alpha-1)}]^{(\alpha-1)}}$$

while for  $0 < \alpha < 1$

$$G_k = \begin{cases} A/C & \gamma_w < |C|^{\alpha} \gamma_{e_k} \\ 0 & \gamma_w > |C|^{\alpha} \gamma_{e_k} \end{cases}$$

$$\gamma_{e_{k+1}} = |B|^{\alpha} \gamma_u + \begin{cases} |A/C|^{\alpha} \gamma_w & \gamma_w < |C|^{\alpha} \gamma_{e_k} \\ |A/C|^{\alpha} |C|^{\alpha} \gamma_{e_k} & \gamma_w > |C|^{\alpha} \gamma_{e_k} \end{cases}$$

*Proof.* The method of proof is straightforward. The basic idea is to minimize the error dispersion for each successive datum or observation as much as possible and thus minimize the pointwise error dispersion. The question of whether or not there exists a solution to this problem, and whether it is unique, can be handled by exhibiting the unique solution explicitly. The error dispersion is simply given by

$$\gamma_{e_{k+1}} = |A - G_k C|^{\alpha} \gamma_{e_k} + |B|^{\alpha} \gamma_u + |G_k|^{\alpha} \gamma_w.$$

Two cases arise. For  $1 < \alpha < 2$ , this is a convex function of  $G_k$ ; thus it is necessary to compute the derivative of  $\gamma_{e_{k+1}}$  with respect to  $G_k$ , set it equal to zero, and solve for the resulting recursive gain. For  $0 < \alpha < 1$ , the error dispersion is not a strictly convex function of  $G_k$ , and we must examine the extreme points of the interval  $(0, A/C)$ . Depending on the values of  $\gamma_w, \gamma_u$ , and  $|C|$ , one or the other of these extrema will minimize the error dispersion.

Having solved for the recursive gain for one particular value of  $k$ , the result is substituted into the recursion for the error dispersion and the process is repeated. The results are summarized in the above formulas.

Q.E.D.

Note that the recursive gain sequence  $\{G_k\}$  is a continuous function of the characteristic exponent  $\alpha$  in the neighborhood of  $\alpha=2$ , the Gaussian case. Preliminary evidence strongly indicates that nonlinear filtering offers performance superior to linear filtering, as seen by [13] and the references therein. We leave that subject for future work.

## APPENDIX

### A BRIEF SUMMARY OF PROPERTIES OF STABLE DISTRIBUTIONS

Suitable references on stable distributions are Feller [14] or Gnedenko and Kolmogorov [15] from which this material is taken.

### Definition.

A probability distribution  $P(x)$  is said to be stable if for all  $a_1 > 0, b_1, a_2 > 0, b_2$  there exists constants  $a > 0, b$  such that

$$P(a_1x + b_1) * P(a_2x + b_2) = P(ax + b).$$

### Theorem

The characteristic function of a stable distribution can be put in the following canonical form

$$E(e^{i\omega x}) = \int_0^{\infty} e^{i\omega x} dP(x) = \exp(i\omega\delta)$$

$$\begin{cases} \exp(-\gamma|\omega|^\alpha [1 + i\beta \operatorname{sign}(\omega)]), & 0 < \alpha < 2, \alpha \neq 1 \\ \exp(-\gamma|\omega| [1 + i\beta \operatorname{sign}(\omega) \ln(\gamma|\omega|)]), & \alpha = 1 \end{cases}$$

where  $-1 < \beta < 1, \gamma > 0, -\infty < \delta < \infty$ .

The parameter  $\alpha$  is called the characteristic index of the stable distribution  $P(x)$ : for  $-1 < \beta < 1$

$$\lim_{x \rightarrow \infty} |x|^\alpha P(-x) = k_- > 0 \quad \lim_{x \rightarrow \infty} |x|^\alpha [1 - P(x)] = k_+ > 0$$

From this it is easy to see that  $E(|x|^p)$  is finite for  $p < \alpha, -1 < \beta < 1$ , or, in particular, that for  $0 < \alpha < 1$  stable distributions possess no mean, while for  $1 < \alpha < 2$  stable distributions possess a mean but no variance. The parameter  $\beta$  is associated with the skewness of the distribution: when  $\beta = 0$ , the distribution is symmetric about  $x = \delta$ , while for  $\beta \neq 0$

$$\lim_{x \rightarrow \infty} \frac{P(-x)}{1 - P(x)} = \frac{1 + \beta}{1 - \beta}$$

$\gamma$  is associated with the dispersion of the distribution; when  $\alpha = 2$  we see that  $\gamma$  equals twice the variance of the distribution.  $\delta$  is a centering or location parameter; for  $\alpha > 1$  the location parameter is also the mean.

Only three analytic closed-form expressions for stable probability-density functions are known at present; the Gaussian ( $\alpha = 2$ ), the Cauchy ( $\alpha = 1, \beta = 0$ ), and the Pearson  $V$  ( $\alpha = \frac{1}{2}, \beta = -1$ ). Otherwise, one must resort to series expansions [14].

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