THE ONSET SPECTRUM OF TURBULENCE

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A general theory for the fluctuation spectrum of the onset of turbulence is developed, applying to systems that approach turbulence through a cascade of subharmonic bifurcations. Applied to Rayleigh–Bénard flow, we find excellent quantitative agreement with the recent experimental data of Libchaber and Maurer.

Until recent times, the onset of turbulence was viewed as a cascade of anharmonic modes leading to a quasi-periodic motion of the fluid (Landau's theory) [1]. More recently modern genericity arguments have supplanted this infinite-cascade picture with one in which two or three such modes initially appear followed by an abrupt transition to turbulent behavior (Ruelle and Takens [2,3]). The newer theory received a measure of experimental support several years ago [4,5]. However, a newly completed experiment of higher resolution [6] in the case of Rayleigh–Bénard flow has now discovered that after two such Landau bifurcations, a frequency-locking phenomenon occurs producing subharmonics of successive half orders of the original frequency. This subharmonic production appears to issue directly into the turbulent regime. Accordingly, neither of the previous pictures correctly describes this kind of onset phenomenon. In this letter we explore a new picture directly rooted in the subharmonic production – thus, we do not consider how the first few bifurcations have led into this regime. However, from this point onwards the theory holds throughout the onset regime. Moreover, unlike the previous theories, the fluctuation spectrum here is quantitatively determined.

The theory to be described leans heavily upon a mathematical framework constructed in recent years [7–10]. We first briefly outline some relevant results.

Consider a one-parameter family of maps on an interval $x_{n+1} = f(\lambda, x_n)$, which for fixed λ possess a unique extremum at a point $\hat{x}$ in the interval. For a large class of such functions, it turns out that as the parameter is varied, the recursion successively possesses stable periodic cycles of order 2^n, with n diverging as the parameter approaches a critical value. Denoting by λ_n that value of parameter at which a bifurcation to a 2^n-cycle has just occurred, the theory determines that

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n)/(\lambda_{n+2} - \lambda_{n+1}) = \delta = 4.6642016 \ldots . \quad (1)$$

As a 2^n-cycle loses stability, after 2^n iterations a point of the attractor just misses duplicating itself, with duplication occurring only after another 2^n iterations. Thus, each element of the cycle splits into closely spaced pairs, with 2^n iterations required to visit an element from its adjacent neighbor. About $\hat{x}$, the theory also determines that from one bifurcation to the next, separation of adjacent elements in a pair is reduced by a universal factor of $-\alpha$ with $\alpha = 2.5029\ldots$.

Generically the extremum is quadratic, so that the images under f of a cluster of points about $\hat{x}$ must rescale as α^2. Indeed, the elements of a cycle in first approximation are of two classes, with half scaling by $-\alpha$ and the other half by α^2.

We need the theory for N-dimensional maps, for $N \gg 1$. In fact a ray in the N-dimensional space is singled out along which the one-dimensional theory holds, thus allowing the theory to be generally applicable [‡1].

[‡1] I have learned of these ideas from J.-P. Eckmann. I have also just learned from P. Collet that a paper is in draft by these two authors providing a rigorous proof.

So far we have discussed maps. In fact, the Fourier-analyzed Navier–Stokes equations possess a "Poincaré map" whose behavior determines that of the flow. Indeed, a host of numerical experiments on the Lorenz model [11], a five-mode truncation of Navier–Stokes in two spatial dimensions [12], and a variety of oscillators have all within the last half year been found to exhibit the successive half subharmonics, convergence rate and rescaling parameter dictated by the one-dimensional mapping theory. We now *conjecture* that the one-dimensional theory holds for the higher order of truncation of Navier–Stokes in three dimensions that presumably dictates the Bénard flow experimentally measured.

Experiment shows that as the heat flux across the convective cells increases, subharmonics of order 2^n are successively generated. With flux value measured as λ_n at the nth bifurcation point, we immediately predict a geometrical convergence at rate δ according to eq. (1). If possible, the measurement of δ would be a definitive test of our theory. (It is to be commented that convergence of δ to several significant figures typically occurs for $n \sim 3,4$.)

In the absence of this test, we next explore the scaling predictions. Construct the map as follows: pick a value of time and locate the trajectory point. Its image is the location of the trajectory point T_0 seconds later where T_0 is the period of basic frequency. Thus in the 2^n subharmonic regime, the map point returns to its initial location after $T_n \equiv 2^n T_0$, and the map has a 2^n-cycle. Starting at a given point y of the cycle, 2^{n-1} iterations will map it to z, the nearest distinct point to y. After one iteration (T_0 later) both points are again nearest elements y', z' at some new location, with arcs of the trajectory linking y and y' and z and z'. If y is in that portion of the cycle such that pairs each rescale by $-\alpha$, then by continuity of the differential flow, the distance between the trajectory arcs must similarly rescale uniformly along their lengths. Thus by appropriate choice of phase along the trajectory, we have the formula

$$\begin{aligned}\psi^{(n+1)}(t) &\equiv [x^{(n+1)}(t) - x^{(n+1)}(t+T_n)] \\ &\approx \alpha^{-1}[x^{(n)}(t) - x^{(n)}(t+T_{n-1})]\,, \quad 0<t<T_{n-1}\,, \\ &\approx \alpha^{-2}[x^{(n)}(t) - x^{(n)}(t+T_{n-1})]\,, \quad T_{n-1}<t<T_n\,,\end{aligned} \tag{2}$$

where $x^{(n)}(t)$ is the coordinate of the trajectory point at the 2^n bifurcation. (Formula (2) is actually a first approximation, with small determined corrections here ignored as negligible.) We now construct the subharmonic spectrum of x from eq. (2).

By the definition of the kth Fourier component of the 2^{n+1}-cycle, $x^{(n+1)}(t)$,

$$x_k^{(n+1)} \equiv \int_0^{T_n} \frac{dt}{2T_n}[x^{(n+1)}(t) + (-1)^k x^{(n+1)}(t+T_n)]\exp(-\pi i k t/T_n)\,. \tag{3}$$

(Observe that the component at the basic period is written as the 2^n harmonic of the *fundamental* of the 2^n-cycle. Thus, the even component $x_{2k}^{(n+1)}$ is at the same frequency as $x_k^{(n)}$.) For k even, since

$$x^{(n+1)}(t) \approx x^{(n+1)}(t+T_n) \approx x^{(n)}(t)\,, \qquad x_{2k}^{(n+1)} \approx x_k^{(n)}\,. \tag{4}$$

Eq. (4) simply says that the component at a given frequency, once it has come into existence for some n, remains approximately unchanged during further bifurcations. The serious calculation is that of $x_{2k+1}^{(n+1)}$, i.e., the spectrum newly introduced with the next bifurcation. By eqs. (2) and (3)

$$x_{2k+1}^{(n+1)} \approx -\frac{1}{\alpha}\int_0^{T_{n-1}} \frac{df}{2T_n}[x^{(n)}(t) - x^{(n)}(t+T_{n-1})]\exp[-\pi i(2k+1)t/T_n]$$

$$+\frac{1}{\alpha^2}\int_{T_{n-1}}^{T_n} \frac{df}{2T_n}[x^{(n)}(t) - x^{(n)}(t+T_{n-1})]\exp[-\pi i(2k+1)t/T_n]\,.$$

Shifting the second integral, together with some manipulation produces

$$x_{2k+1}^{(n+1)} \approx \frac{1}{2\alpha}\left[1 - i\frac{(-1)^k}{\alpha}\right] \int_0^{T_{n-1}} \frac{dt}{2T_{n-1}} \left[x^{(n)}(t) - x^{(n)}(t \mp T_{n-1})\right] \exp\left(-\pi i \frac{2k+1}{2} t/T_{n-1}\right) . \tag{5}$$

To proceed, substitute for $x^{(n)}$ its Fourier expansion and obtain

$$x_{2k+1}^{(n+1)} \approx \frac{1}{2\alpha\pi i} \left[1 - i(-1)^k\right] \left[1 - i\frac{(-1)^k}{\alpha}\right] \sum_{k'} \frac{1}{(2k'+1) - \frac{1}{2}(2k+1)} x_{2k'+1}^{(n)} . \tag{6}$$

Eq. (6) is the basic result of the theory. Observe that the new fluctuation components $x_{2k+1}^{(n+1)}$ are determined solely by what had been the new components, $x_{2k'+1}^{(n)}$: the basic cycle's fundamental $x^{(n)}{}_{2n}$ and its harmonics determining the shape of the "basic" cycle are all *even* harmonics and so *decouple* from the fluctuation spectrum. Thus the fluctuation spectrum itself has a character universal over the specific form endowed by a particular system upon its basic cycle. Partially compromising this result is the fact that eq. (6) is exact only for n asymptotically large – or for several significant figures, after the first several subharmonic bifurcations.

Since the phase is critical in employing eq. (6), we derive its asymptotic form for the amplitude spectrum. Setting $2k + 1 \equiv \xi$, $2k' + 1 \equiv \xi'$, in the limit of large n,

$$|x^{(n+1)}(\xi)| \approx (4\alpha)^{-1} \left[2(1 + 1/\alpha^2)\right]^{1/2} \left| \frac{P}{\pi i} \int_{-\infty}^{+\infty} \frac{d\xi'}{\xi' - \xi/2} x^{(n)}(\xi') \right| . \tag{7}$$

By eq. (3) $x_k^{(n)}$ continues for large T_n to $x^{(n)}(\xi)$ with $x^{(n)}$ analytic in the lower half plane, so that by a Hilbert transform result

$$|x^{(n+1)}(\xi)| \approx (4\alpha)^{-1} \left[2(1 + 1/\alpha^2)\right]^{1/2} |x^{(n)}(\xi/2)| . \tag{8}$$

According to eq. (8), to obtain the new spectral components interpolate on a smooth fit to the previously new components, and rescale by the *universal* factor

$$\mu \equiv 4\alpha/\left[2(1 + 1/\alpha^2)\right]^{1/2} = 6.57, \quad \text{or} \quad 10 \log_{10}\mu = 8.18 \text{ db} .$$

Observe that eq. (2) is a *vector* equation in N-dimensions. Since the Fourier manipulations are all linear, eq. (6) is correct for *each* component (i.e., *spatial* Fourier mode) or any linear combination of components (e.g. the time spectral coefficients of the velocity field at each point in the fluid). Thus the time spectra of the temperature pressure and velocity fields at any point in the fluid satisfy eq. (6), or approximately eq. (8): given the spectrum of any such observable down to the 2^k subharmonics, the rest of the fluctuation spectrum is computable. In particular, we consider the experimental spectrum for the temperature of a Rayleigh–Bénard flow, fig. 1. Now the deduction of eq. (8) contains a principal error for finite n consisting of an oscillation riding on top of the smooth extrapolation, of amplitude determined by the *phased* components. Accordingly, eq. (8) determines the new components with these oscillations smoothed away. Starting at $n = 2$ as "asymptotic" the interpolated curve of the 1/4 and 3/4 components is a horizontal line. Accordingly, by eq. (8) the $n = 3$ components should lie about a horizontal line 8.2 db lower, and the $n = 4$ components about another line 8.2 db still lower. By fig. 1, the $n = 3$ components are 8.4 ± 0.5 db down from $n = 2$. For $n = 4$ the lowest components are missing, but can be projected to have less oscillation than the higher ones, while the highest ones are least well predicted. We approximate this ignorance by deleting the highest component, resulting in the average $n = 4$ component 8.3 ± 0.4 db below the $n = 3$. Thus, eq. (8) is satisfied to within experimental and theoretical error, and we interpret this agreement to be the signal that the system accomplishes its transition to turbulence through the fixed point of the universality theory.

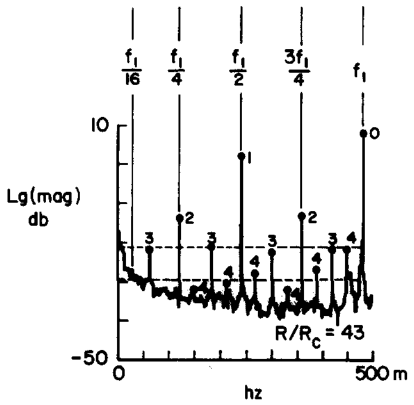

Fig. 1. (Redrawn from ref. [6].) Points are labeled with n (see text). The dashed horizontal lines are the theoretical averages for n = 3, 4.

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